1. For $a, b \in \mathbb{N}$, prove that $\gcd(b, a) = \gcd(b, a \bmod b)$.

2. Suppose $a \equiv b \mod m$ and $c \equiv d \mod m$. Let $n \in \mathbb{Z}_+$. Prove the following two statements:

   (a) $ac \equiv bd \mod m$

   (b) $a^n \equiv b^n \mod m$

3. For any $n \in \mathbb{Z}_+$, define the relation R_n on $\mathbb{Z}$ by

$$R_n = \{(x, y) \mid x \text{ and } y \text{ have the same remainder when divided by } n\}.$$

   Prove that R_n is an equivalence relation.

4. Let a and b be integers, not both zero. Prove that the smallest positive integer of the form $ax + by$,where x and y are integers, is $\gcd(a, b)$.

5. Consider the two pairs ($a = 16133$, $b = 5814$) and ($a = 128374$, $b = 59345$). For each pair, do the following:

   (a) Compute $\gcd(a, b)$.

   (b) Find values of x and y such that $ax + by = \gcd(a, b)$.

   (c) Determine whether a and b are relatively prime. If they are, find the modular (multiplicative) inverse of b mod a. In other words, find c such that $bc \equiv 1 \bmod a$.

6. In $\mathbb{Z}_{64}$, find $3 \oplus 51, 3 \ominus 51, 3 \otimes 51$, and $3 \oslash 51$.

7. Prove that consecutive integers must be relatively prime.

8. Let $a \in \mathbb{Z}$. Prove that $2a + 1$ and $4a^2 + 1$ are relatively prime.

9. Prove that among any three distinct integers we can find two, say a and b, such that the number $a^3b - ab^3$ is a multiple of 10.

10. Compute $2^{25} + 3^{26} \bmod 11$.

11. Find the last two digits of $7^{7^{7^7}}$.

12. An old woman went to the market and a horse stepped on her basket and smashed her eggs. The rider offered to pay for the eggs and asked her how many there were. She did not remember the exact number, but when she had taken them two at a time there was one egg left, and the same happened when she took three, four, five, and six at a time. But when she took them seven at a time, they came out even. What is the smallest number of eggs she could have had?