PILOT

Discrete Math
Problem-Set #7
Spring 2019

1. Which of the following equations are functions?
   (a) $x + y = 64$
   (b) $x^2 + y^2 = 64$
   (c) $x^3 + y^3 = 64$
   (d) $|y| - |x| = 3$
   (e) $y = e^x$
   (f) $y = \sin x$

2. For the equations that are functions, which are one-to-one or onto?

3. Which functions above have an inverse function? What does a function need to be to have an inverse function?

4. Let $A = \{1, 2, 3\}$. Determine if the following relations are functions:
   (a) $R1 = \{(1, 2), (2, 1)\}$.
   (b) $R2 = \{(1, 2), (2, 1), (3, 2)\}$.

5. Determine if the following functions are injective, surjective, or neither. Also state if they are bijective.
   (a) $f : \{0, 1\} \to \mathbb{N}$ where $f(0) = 1$ and $f(1) = 0$.
   (b) $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^2$.
   (c) $f :$ JHU students $\to$ Countries in the world where $f(\text{student}) =$ country where student is from.
   (d) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$.
   (e) $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = \frac{x}{2}$.

   Note: $\mathbb{N} = \{0, 1, 2, \ldots\}$

6. Let $A = \{0, 1\}$ and $B = \{b : b \subseteq A \times A\}$. Let F be the set of all possible functions from A to A.
   (a) Does there exist an injection from F to B? If so, give an injection and prove it is indeed an injection. If not, prove why such a function does not exist.
   (b) Does there exist a surjection from F to B? If so, give a surjection and prove it is indeed a surjection. If not, prove why such a function does not exist.
   (c) Does there exist a bijection from F to B? Why or why not?

7. Two sets A and B are said to have the same cardinality (same size $\Leftrightarrow |A| = |B|$), if and only if we can construct a bijection from $A \to B$ (or, equivalently, from B to A). Let E be the set of even natural numbers, that is, $E = \{0, 2, 4, \ldots\}$ and O be the set of odd natural numbers, $O = \{1, 3, 5, \ldots\}$. Show the following:
   (a) $|E| = |O|$.
   (b) $|E| = |\mathbb{N}|$.

8. Let A be a set with finite cardinality $n > 0$.
   (a) Prove that the number of subsets of A of size k is equal to the number of subsets of A of size $n - k$. *Hint:* Construct a bijection (be sure to show your function is indeed a bijection).
   (b) Prove that the number of odd-cardinality subsets of A is equal to the number of even-cardinality subsets of A. *Hint:* Construct a bijection (be sure to show your function is indeed a bijection).

Post Spring Break Check-in

- *Now that the semester is about half-way over, how are you feeling going into the 2nd half?*
- *Are you going to class?*
- *What are your study habits like? How are you preparing for exams? Any tips that have worked with you to share with the group?*
- *What do you do for yourself – what do you do to relieve stress?*

9. For $a, b \in \mathbb{N}$, prove that $\gcd(b, a) = \gcd\ (b, a \bmod b)$.

10. Suppose $a \equiv b \mod m$ and $c \equiv d \mod m$. Let $n \in \mathbb{Z}_{+}$. Prove the following two statements:

    (a) $ac \equiv bd \mod m$

    (b) $a^n \equiv b^n \mod m$

11. For any $n \in \mathbb{Z}_{+}$, define the relation R_n on $\mathbb{Z}$ by

$$R_n = \{(x, y) \mid x \text{ and } y \text{ have the same remainder when divided by } n\}.$$

    Prove that R_n is an equivalence relation.

12. Let a and b be integers, not both zero. Prove that the smallest positive integer of the form $ax + by$,where x and y are integers, is $\gcd(a, b)$.

13. Consider the two pairs ($a = 16133$, $b = 5814$) and ($a = 128374$, $b = 59345$). For each pair, do the following:

    (a) Compute $\gcd(a, b)$.

    (b) Find values of x and y such that $ax + by = \gcd(a, b)$.

    (c) Determine whether a and b are relatively prime. If they are, find the modular (multiplicative) inverse of b mod a. In other words, find c such that $bc \equiv 1$ mod a.