PILOT

Discrete Math
Problem-Set #6
Spring 2019

1. Prove that if n^2 is an even integer, then n is even.
2. Let $x \in \mathbb{Z}$. Prove that if $x^2 - 6x + 5$ is even, then x is odd.
3. If $x^5 + 5 < 0$, prove that $x < 0$.
4. Let $a, b, n \in \mathbb{Z}$. If $n \nmid ab$ then show that $n \nmid a$ and $n \nmid b$.
5. Show that for all $n \in \mathbb{Z}$, that $3n + 2$ is never a perfect square.
6. (a) Let n be a nonnegative integer integer. Show that $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
   (b) Show that $2^n > n$ for all positive integers n.
   (c) Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfying $f(a+b) = f(a) + f(b)$. Show that $f(n) = nf(1)$ for all nonnegative integers n.
7. (a) A real number is said to be rational if it can be written in the form $\frac{a}{b}$ where a, b are integers and $b \neq 0$. A real number that is not rational is irrational. Show that $\sqrt{2}$ is irrational.
   (b) Prove that the sum of a rational number and an irrational number is irrational.
   (c) Suppose that 5 points are placed inside of an equilateral triangle that has side lengths of 1. Show that there exists a pair of points whose distance is no more than $\frac{1}{2}$.
8. Prove that there are infinite prime numbers.
9. Show that the sum of any four consecutive integers is not divisible by 4.
10. Show that no set of nine consecutive integers can be partitioned into two sets with the product of the elements of the first set equal to the product of the elements of the second set.
11. Show that $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.
12. Prove that $3^n \geq n^3$ for all positive integers n.
13. Prove that a square can be partitioned into n smaller squares for any $n \geq 4$.
14. Which of the following equations are functions?
    (a) $x + y = 64$
    (b) $x^2 + y^2 = 64$
    (c) $x^3 + y^3 = 64$
    (d) $|y| - |x| = 3$
    (e) $y = e^x$
    (f) $y = \sin x$
15. For the equations that are functions, which are one-to-one or onto?
16. Which functions above have an inverse function? What does a function need to be to have an inverse function?
17. Let $A = \{1, 2, 3\}$. Determine if the following relations are functions:
    (a) $R1 = \{(1, 2), (2, 1)\}$.
    (b) $R2 = \{(1, 2), (2, 1), (3, 2)\}$.